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CMDH/M-24 **5521**

MECHANICS AND CALCULUS OF VARIATIONS

Paper–MMATH 21–401

Time Allowed : 3 Hours] [Maximum Marks : 80

Note : Attempt **five** questions in all, selecting **one** question from each Unit. Question No. **9** is compulsory. All questions are carry equal marks.

UNIT–I

1. (a) Derive the equation for vector angular momentum : 8

$\vec{H} = A^* w^{(1)} + B^* w^{(2)} + C^* w^{(3)}$ A^*, B^*, C^* are the principle moments of inertia about x, y, z-axes.

(b) For general motion of a rigid body prove the relations : 8

(i) $T = \frac{1}{2} M v^{-2} + \frac{1}{2}\left(A^* w_1^2 + B^* w_2^2 + C^* w_3^2\right).$

(ii) $\vec{H}' = A^* w_1 \, \hat{a}_1 + B^* w_2 \, \hat{a}g + C^* w_3 \, \hat{a}_3.$

2. (a) A uniform circular disc of mass M and radius 'a' is rotating in it plane with initial angular velocity w, its centre being at rest. If a point on the rim be suddenly fixed, find the new angular velocity of the disc and the velocity of its centre. 8

(b) A uniform heavy solid hemisphere of radius 'a' is held at rest with its base vertical and its curved surface in contact with a horizontal plane. If the hemisphere is released when the plane is rough enough to prevent slipping, show that the angle θ that the base makes with the horizontal at time 't' is such that $\left(\frac{d\theta}{dt}\right)^2 = \frac{15g\cos\theta}{\left[a\left(28-15\cos\theta\right)\right]}.$ 8

UNIT–II

3. (a) A rigid body is free to rotate about its centroid G, the principal moments of inertia at which are 7, 25, 32 units respectively. The body is given an angular velocity Ω about a line through G whose direction ratios are 4 : 0 : 3. Show that after time 't' the components of angular veolcity about the principal axes of inertia at G are $\frac{4}{5}\Omega\cos\phi, \frac{4}{5}\Omega\sin\phi, \frac{3}{5}\Omega\cos\phi$

where, $\tan\left(\frac{\phi}{2}\right) = \tanh\left(\frac{3\Omega t}{10}\right).$

Deduce that ultimately the body rotates about the principal axis of intermediate moment of inertia. 8

(b) Derive the Euler's dynamical equations for the motion of a rigid body about a fixed point and hence prove the relations : 8

(i) $Aw_1^2 + Bw_2^2 + Cw_3^2 = \text{constant.}$

(ii) $A^2w_1^2 + B^2w_2^2 + C^2w_3^2 = \text{constant.}$

4. (a) Derive Lagrange's equations for impulsive forces. 8

(b) Two uniform rods AB, BC of masses m_1, m_2 and lengths $2a$, $2b$ are smoothly hinged at B and initially they lie at rest on a smooth table and in a straight line. AB receives a blow of impulse I at A perpendicular to AB. Construct the equations of motion of the system just after impact. 8

UNIT–III

5. (a) Derive Hamilton's canonical equations : 8

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \text{ and } \frac{dp_i}{dt} = \frac{-\partial H}{\partial q_i}.$$

(b) Prove that the formula for variation of action is : 8

$$\delta w = \left[\sum_{i=1}^{n} p_i \delta q_i - H \delta t \right]_{t_0}^{t_1}$$

6. (a) If $s(t, q_i, \alpha_i)$ is source complete integral of the Hamilton-Jacobi equation $\frac{\partial s}{\partial t} + H\left(t, q_i, \frac{\partial s}{\partial q_i}\right) = 0$, then the final equation of motion of a holonomic system with the given function H may be written in the form $\frac{\partial s}{\partial q_i} = p_i, \frac{\partial s}{\partial \alpha_i} = \beta_i = 1 + i + 0\, n \quad \alpha_i, \beta_i$ are arbitrary constants. 8

(b) Show that the transformation $\tilde{q} = \sqrt{2q} \cos p, \tilde{p} = \sqrt{2q} \sin p$ is a canonical transformation. 8

UNIT–IV

7. (a) Find the curve passing through (x_0, y_0) and (x_1, y_1) which generates the surface of minimum area when rotated about the $x - axis$. 8

(b) Find the extremal of the functional $I[y(x), z(x)] =$

$$\int_0^1 \left(y'^2 + z'^2\right) dx \text{ with } \begin{array}{l} y(0) = 0, z(0) = 0 \\ y(1) = 1, z(1) = 2 \end{array}$$ 8

8. For a functional :

(a) $I[y(x)] = \int_a^b f(x, y, y')\, dx$, if $y(x)$ is the extremal there obtain the natural boundary conditions. 8

(b) Find the distance between the parabola $y = x^2$ and the straight line $x - y = 5$. 8

Compulsory Question

9. Attempt the following questions : 8×2=16

(i) If $F(x, y, y') = M(x, y) + N(x, y)y'$ then show that $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$ is the Euler's equations.

(ii) State Brachistochrone Problem.

(iii) State Perpendicular Axes theorem.

(iv) If $\vec{H} = \frac{2}{5} Ma^2 \left(w_1 \hat{i} + w_2 \hat{j} + w_3 \hat{k} \right)$ and $\vec{W} - w_1 \hat{i} + w_2 \hat{j} + w_3 \hat{k}$

then find $\frac{d\vec{H}}{dt} = ?$

(v) Define Holonomic and Scleronomic systems.

(vi) Obtain general equation of Dynamics.

(vii) Define poisson Bracket for the functions ϕ and ψ.

(viii) Show that the transformation an $\tilde{q}_i = \alpha p_i$ and $\tilde{p}_i = \beta q_i$ is canonical with $\tilde{H} = -\alpha\beta H$.