(b) If p is a polynomial in λ with real coefficients, say,

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \ldots\ldots + \alpha_0,$$

then the operator p(T) defined by

$$p(T) = \alpha_n T^n + \alpha_{n-1} T^{n-1} + \ldots\ldots + \alpha_0 I$$

has the spectral representation

$$p(T) = \int_{m-0}^{M} P(\lambda)\, d E_\lambda$$

and for all $x, y \in H$,

$$< p(T) x, y > = \int_{m-0}^{M} p(\lambda) dw(\lambda).$$

UNIT–IV

8. (a) State and prove Hellinger - Toeplitz theorem. 6

(b) Let $T : D(T) \to H$ be a linear operator where H is a complex Hilbert space and D(T) is dense in H. Show that if T is symmetric then its closure $\overline{T}$ exists and is unique. 10

9. (a) Show that the Cayley transform $U = (T - iI)(T + iI)^{-1}$ of a self-adjoint linear operator $T : D(T) \to H$ exists on H and is a unitary operator, where, $H \neq \{0\}$ is a complex Hilbert space. 6

(b) Let T be the multiplication operator and $\sigma(T)$ its spectrum. show that : 10

(i) T has on eigen values.

(ii) $\sigma(T)$ is all of $\mathbb{R}$.

CMDQ/M-24 **5528**

ADVANCE FUNCTIONAL ANALYSIS

Paper–MMATH 21–410

Time allowed : 3 Hours] [Maximum Marks : 80

Note : Attempt **five** questions in all, selecting **one** question from each unit. Question No. **1** is compulsory. All questions carry equal marks.

Compulsory Question

1. Attempt all questions: $8 \times 2 = 16$

(i) State (only) spectral radius formula.

(ii) Show that for a given linear operator T, the sets $\sigma_p(T)$ and $\sigma_c(T)$ are disjoint.

(iii) Show that every linear operator on a finite dimensional space is compact.

(iv) Explain the term Fredholm alternative.

(v) If two bounded self adjoint linear operators S and T on a Hilbert space are positive and ST = TS, show that ST is positive.

(vi) Show that the sum of two projections need not be a projection.

(vii) State (only) Wecken's lemma.

(viii) Define: (a) Multiplication operator

(b) Differentiation operator.

UNIT–I

2. State and prove spectral mapping theorem for polynomials. 16

3. (a) If $X \neq \{0\}$ is a complex Banach space and $T \in B(X, X)$, show that $\sigma(T) \neq \phi$. 10

   (b) Let A be a complex Banach algebra with identity *e*. Show that for any $x \in A$, the spectrum $\sigma(x)$ is compact, and the spectral radius satisfies $r_\sigma(x) \leq \| x \|$. 6

UNIT–II

4. (a) Show that the range $\mathbb{R}(T)$ of a compact linear operator $T : X \rightarrow Y$ is separable, where, X and Y are normed spaces. 6

   (b) Let $T : X \rightarrow Y$ be a linear operator. If T is compact, show that its adjoint operator $T^x : Y' \rightarrow X'$ where X and Y are normed spaces, is also compact. (X' and Y' are the dual spaces if X and Y respectively). 10

5. (a) Let $T : X \rightarrow X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Show that there exists a smallest integer *r* (depending on λ) such that from $n = r$ on, that null spaces $N(T_\lambda^{\,n})$ are all equal, and if $r > 0$, the inclusions

   $N(T_\lambda^{\,0}) \subset N(T_\lambda) \subset \ldots \subset N(T_\lambda^{\,r})$ are all proper. 10

   (b) Let $T : X \rightarrow X$ be a compact linear operator on a normal space X. If T has non zero spectral values, show that every one of them must be an eigen value of T. 6

UNIT–III

6. (a) Let $T : H \rightarrow H$ be a bounded self–adjoint linear operator on a complex Hilbert space H. Show that a number $\lambda \in e(T)$, the resolvent set of T, if and only if there exists a $c > 0$ such that for every $x \in H$, $\| T_\lambda x \| \geq C \| x \|$ $(T_\lambda = T - \lambda I)$. 10

   (b) Show that the spectrum $\sigma(T)$ of a bounded linear operator $T : H \rightarrow H$ on a complex Hilbert space H lies in the closed interval [m, M] on the real axis, where

   $$m = \inf_{\| x \| = 1} < Tx, x >, M = \sup_{\| x \| = 1} < Tx, x >. \qquad 6$$

7. Let $T : H \rightarrow H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Show that: 16

   (a) T has the spectral representation

   $$T = \int_{m-0}^{M} \lambda dE_\lambda$$

   where $E = (E_\lambda)$ is the spectral family associated with T, and for all $x, y \in H$,

   $$< Tx, y > = \int_{m-0}^{M} \lambda dw(\lambda), \; w(\lambda) = < E_\lambda x, y >.$$