Roll No. .... Total Pages : 3

CMDQ/M-24 **5524**

ALGEBRAIC NUMBER THEORY

Paper–MMATH 21–406

Time Allowed : 3 Hours] [Maximum Marks : 80

Note : Attempt **five** questions in all, selecting **one** question from each Unit. Question No. **1** is compulsory. All questions carry equal marks.

Compulsory Question

1. Answer the following :

   (a) Let $k = \mathbb{Q}(i)$. Show that $i \in O_k$ and $N_k(i) \in \mathbb{Z}$.

   (b) Define discriminant of an algebraic number field k.

   (c) Define Neotherian ring.

   (d) State Chinese remainder theorem.

   (e) When two fractional ideals in k are said to be equivalent ?

   (f) Show that relation defined in part (e) is an equivalence relation.

   (g) Define Lgendre Symbol $\left(\frac{a}{p}\right)$, with p prime.

   (h) If $a \equiv b \pmod{p}$, prove that $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

UNIT–I

2. Let $m \in \mathbb{Z}$, $\alpha \in O_k$ and let $f(x)$ be the minimal polynomial of α. Prove that

$$d_{k/\mathbb{Q}}(\alpha + m) = (-1)^{{}^{c}n_2} \prod_{i=1}^{n} f^1\left(\alpha^{(i)}\right).$$

3. Let $k/\mathbb{Q}$ be an algebraic number field of degree n. Prove that $d_k \in \mathbb{Z}$ and $d_k \equiv 0$ or 1(mod 4).

UNIT–II

4. Show that any fractional ideal $A \neq 0$ can be written uniquely in the form

$$\frac{{}_1\ {}_2 \cdots\cdots {}_r}{{}_1^1\ {}_2^1 \cdots\cdots {}_s^1} \text{ where } {}_i \text{ and } {}_i^1$$

may be repeated, but no ${}_i = {}_i^1$

Also prove that there exists a fractional ideal $\mathbb{A}^1$ such that $\mathbb{A}\mathbb{A}^1 = O_k$.

5. Show that D^{-1} is a fractional ideal of k and find an integral basis. Also prove that D is an ideal of O_k is D is fractional ideal-inverse of D^{-1}.

UNIT–III

6. Let k > 0 be a squarefree positive integer. Suppose that $k \equiv 1, 2 \pmod 4$, and k does not have the form $k = 3a^2 \pm 1$ for an integer a. Consider the equation $x^2 + k = y^3$.

Show that of 3 does not divide the class number of $\mathbb{Q}\left(\sqrt{-k}\right)$, then this equation has no integral solution.

7. Compute the class number of fields $\mathbb{Q}\left(\sqrt{-1}\right)$, $\mathbb{Q}\left(\sqrt{-2}\right)$, $\mathbb{Q}\left(\sqrt{-3}\right)$ and $\mathbb{Q}\left(\sqrt{-7}\right)$.

UNIT–IV

8. Compute $\left(\frac{5}{p}\right)$ and $\left(\frac{7}{p}\right)$ if p is an odd prime.

9. Show that there are infinitely many primes of the form 5k + 4 and of the form 15k + 4.